A REVIEW OF NUMERICAL METHODS FOR THE ANALYSIS OF COMPOSITE AND SANDWICH STRUCTURES

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Abstract

In this paper an overview of modelling techniques is presented associated with the finite element method for the analysis of laminated shell structures. A review of the method and its applications in composite and sandwich structures is discussed. To illustrate the particular aspects of the formulations, some theory formulations are considered and explained.

Finite element method

The solution of the governing equations of laminated composite or sandwich beams, plates and shells can be solved by analytical methods, such as the Navier, Lévy or Rayleigh-Ritz techniques. However, exact analytical or variational solutions to these problems cannot be developed when complex geometries, arbitrary boundary conditions, or nonlinearities are involved. Therefore, approximate methods of analysis become relevant when there is a need for solving such problems.

The finite element method is a powerful computational technique for the solution of differential and integral equations that arise in various fields of engineering and applied science. The method is a generalisation of the classical variational and weighted-residual methods [1-5]. Since most real problems are defined on domains that are geometrically and boundary complex, it is difficult to generate approximation functions required in traditional variational methods. The basic idea of the finite element method is to view a given domain as an assembled set of simple geometries, called finite elements, for which it is possible to generate the approximation functions needed in the solution of differential equations by any variational or weighted-residual methods. The ability to represent domains with irregular geometries by a collection of finite elements makes the method a valuable tool for the solution of boundary, initial and eingenvalue problems arising in various fields of engineering. The finite element method is an element-wise application of the variational or weighted-residual methods, by the use of approximation functions, called interpolation functions [1-5]. For a particular problem, including laminated composite materials, it is possible to develop special finite element approximations (finite element models), depending on the choice of the variational or weighted-residual methods. The major steps of the finite element method include: the discretisation of the domain into a set of finite elements (mesh); weak formulation (weighted-residuals) of the differential equations over a typical finite element (subdomain); development of the finite element model of the problem using its weak form. The finite element model consists of a set of algebraic equations among the unknown parameters of the element; assembly of the elements to obtain a global system of algebraic equations; imposition of boundary equations; solution of equations; post-computing of solution and quantities of interest [1-5]. This modular approach is interesting for computer programming, allowing the coupling of various physical problems.

Single-layer theories

Composite laminates are formed by stacking layers of different composite materials and /or fiber orientation. By construction, composite laminates have their planar dimensions one or two orders of magnitude larger than their thickness. Often laminates are used in applications that require axial and bending strengths. Therefore composite laminates are treated as plate or shell elements.

The analyses of composite plates and shells in the past have been based on one of the following approaches:

- Equivalent single-layer theories (2-D)
    - Classical laminate theory
    - Shear deformation laminate theories
- Three-dimensional elasticity theory (3-D)
    - Traditional 3-D elasticity formulations
    - Layerwise theories

The equivalent single-layer theories are derived from the 3-D elasticity theory by making suitable assumptions concerning the kinematics of deformation of the stress state through the thickness of the laminate. These assumptions allow the reduction of a 3-D problem to a 2-D problem. In the three-dimensional or layerwise theories each composite layer is treated as a 3-D solid. Relevant publications of the single-layer theories can be studied in [6-15]. As an example of the development of a 2-D first-order shear deformation theory, Yang, Norris and Stavsky [8] have developed a theory of deformation based on the work of Mindlin [6] and Reissner [7] for isotropic plates. The displacement field is obtained as

$$\begin{aligned} u(x,y,z) &= u_o(x,y) + z\theta_x(x,y) \\ v(x,y,z) &= v_o(x,y) + z\theta_y(x,y) \\ w(x,y,z) &= w_o(x,y) \end{aligned} \qquad (1)$$

where u, v and w are the x,y and z displacements. The u_o, v_o and w_o are the mid-plane displacements, and θ_x and θ_y the rotations of the normal. Strain-displacement relations can be expressed as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{bmatrix} + z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \underline{\varepsilon}^o + z\underline{\kappa} \qquad \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} w_{,x} + \theta_x \\ w_{,y} + \theta_y \end{bmatrix} \qquad (2)$$

where

$$\begin{aligned} &\varepsilon_x^o = u_{o,x}\,, \varepsilon_y^o = v_{o,y}\,, \gamma_{xy}^o = v_{o,x} + u_{o,y} \\ &\kappa_x = \theta_{x,x}\,, \kappa_y = \theta_{y,y}\,, \kappa_{xy} = \theta_{x,y} + \theta_{y,x} \end{aligned} \qquad (3)$$

are the membrane deformations and plane curvatures. In each layer k of the laminate the stress-strain relations are obtained as

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\end{bmatrix}_k = \begin{bmatrix}c_{11} & c_{12} & c_{13}\\ c_{12} & c_{22} & c_{23}\\ c_{13} & c_{23} & c_{33}\end{bmatrix}_k \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix}_k$$

$$\begin{bmatrix}\tau_{13}\\ \tau_{23}\end{bmatrix}_k = \begin{bmatrix}c_{44} & c_{45}\\ c_{45} & c_{55}\end{bmatrix}_k \begin{bmatrix}\gamma_{13}\\ \gamma_{23}\end{bmatrix}_k \qquad (4)$$

where c_{ij} are elastic coefficients and {1,2,3} are the principal material directions for each layer. This theory is simple to implement, but introduces a shear correction coefficient that affects only the shear terms.

Refined theories

The classical laminated plates theory and the first-order shear deformation theory are the simplest equivalent single-layer theories, describing adequately the kinematic behaviour of most laminates. The higher order shear deformation theories avoid the use of the shear correction factors. Various theories were proposed in the literature [16-21]. The model proposed in this paper has the following displacement field

$$\underline{u} = \underline{u}^0 + z\underline{\theta} + z^3\underline{\theta}^* \qquad (5)$$

where $\underline{\theta}^*$ represents the higher order rotations. In these theories the transverse displacement is constant throughout the plate or shell thickness. This theory has the advantage that no shear-correction factors are needed. In sandwich laminates these theories present some difficulties.

Layerwise theories

The layerwise theory [22-27] allows for a better deformation analysis in the laminate. In this work it is followed the concepts proposed by Manewya and Davies [26] for laminated plates. This concept was also explored by Al-Quarra [27] and Ferreira et al [23]. In this theory the displacement field for each layer (n) is a function of the displacements of other layers adjacent, as

$$\begin{aligned} u^{(n)} &= u^{(n-1)} + z\theta_x^{(n)}\\ v^{(n)} &= v^{(n-1)} + z\theta_y^{(n)}\\ w^{(n)} &= w_0 \end{aligned} \qquad (6)$$

The theories presented so far were implemented in a degenerated shell element [11-14]. The coordinates **x** of a point within the element are obtained as

$$\mathbf{x} = [x, \mathrm{y}, \mathrm{z}]^T = \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{N_k}(\xi,\eta)\left[\mathbf{x}_k^{mid} + \mathrm{h_k}\ \zeta/2\ \bar{\mathbf{v}}_{3\mathrm{k}}\right] \qquad (7)$$

The displacements for the first order shear deformation are obtained as

$$\underline{u} = \sum_{k=1}^{n} \underline{N}_k\,\underline{u}_k^{med} + \sum_{k=1}^{n} \underline{N}_k\,\zeta\,\frac{h_k}{2}\left[V_{1k}, -V_{2k}\right]\begin{Bmatrix}\beta_{1k}\\ \beta_{2k}\end{Bmatrix} \qquad (8)$$

whereas for the third order theory are obtained as

$$\underline{u} = \sum_{k=1}^{n} \underline{N}_k\underline{u}_k^{med} + \sum_{k=1}^{n} \underline{N}_k\zeta\,\frac{h_k}{2}\left[V_{1k}, -V_{2k}\right]\begin{Bmatrix}\beta_{1k}\\ \beta_{2k}\end{Bmatrix} + \sum_{k=1}^{n} \underline{N}_k\left(\zeta\,\frac{h_k}{2}\right)^3\left[V_{1k}, -V_{2k}\right]\begin{Bmatrix}\beta_{1k}^*\\ \beta_{2k}^*\end{Bmatrix} \qquad (9)$$

In the layerwise theory the displacements are obtained as

$$\underline{u}^{(1)} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^{(1)} = \sum_{k=1}^{n} N_k \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^{méd} + \sum_{k=1}^{n} N_k \zeta \frac{h_k^{(1)}}{2} \left[\underline{V}_{1k}, -\underline{V}_{2k} \right] \begin{Bmatrix} \beta_{1k}^{(1)} \\ \beta_{2k}^{(1)} \end{Bmatrix} \tag{10}$$

for the first layer and

$$\underline{u}^{(n)} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^{(n)} = \sum_{k=1}^{n} N_k \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^{med(n)} + \sum_{k=1}^{n} N_k \zeta \frac{h_k^{(n)}}{2} \left[\underline{V}_{1k}, -\underline{V}_{2k} \right] \begin{Bmatrix} \beta_{1k}^{(n)} \\ \beta_{2k}^{(n)} \end{Bmatrix} \tag{11}$$

where

$$\underline{u}^{med(n)} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^{med(n)} = \sum_{k=1}^{n} N_k \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^{med(n-1)} + \sum_{k=1}^{n} N_k \frac{h_k^{(n-1)}}{2} \left[\underline{V}_{1k}, -\underline{V}_{2k} \right] \begin{Bmatrix} \beta_{1k}^{(n-1)} \\ \beta_{2k}^{(n-1)} \end{Bmatrix} + \sum_{k=1}^{n} N_k \frac{h_k^{(n)}}{2} \left[\underline{V}_{1k}, -\underline{V}_{2k} \right] \begin{Bmatrix} \beta_{1k}^{(n)} \\ \beta_{2k}^{(n)} \end{Bmatrix} \tag{12}$$

for the other layers.

Constitutive relations

The constitutive equations are derived for an orthotropic material with the material axes 1, 2 parallel to the plane x-y and rotated by some angle θ in relation to the axes x and y, being the material axe 3 parallel to z . Taking into account that $\sigma_z = 0$, the stresses σ at a layer of the element are related to the deformations ε by [11-14]

$$\sigma = \left[\sigma_x \ , \ \sigma_y \ , \ \tau_{xy} \ , \ \tau_{xz} \ , \ \tau_{yz} \right]^T = \mathbf{D} \ \varepsilon = \mathbf{T}^T \ \overline{\mathbf{D}} \ \mathbf{T} \ \varepsilon \tag{13}$$

where $\mathbf{T} = \begin{bmatrix} \mathbf{T}_1^{'} & \mathbf{O} \\ \mathbf{O} & \mathbf{T}_2 \end{bmatrix}$ $\quad \mathbf{T}_1^{'} = \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2 - s^2 \end{bmatrix}$ $\quad \mathbf{T}_2 = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$ (14)

is the strain transformation matrix, in which c = cos θ and s = sin θ, and $\overline{\mathbf{D}}$ is the material elasticity matrix, defined for the material axes 1, 2 , 3 , having non-zero terms

$$\overline{D}_{11} = \frac{E_1}{1 - \nu_{12} \nu_{21}} \qquad \overline{D}_{22} = \frac{E_2}{1 - \nu_{12} \nu_{21}} \qquad \overline{D}_{33} = G_{12} \tag{15}$$

$$\overline{D}_{12} = \nu_{12} \overline{D}_{22} \qquad \overline{D}_{44} = k_{13} G_{13} \qquad \overline{D}_{55} = k_{23} G_{23} \tag{16}$$

in which E_1 and E_2 are the Young's moduli in the 1 and 2 directions respectively, ν_{ij} is Poisson's ratio for transverse strain in the i-direction when stressed in the j-direction and G_{13}

and G_{23} are the shear moduli in the 1-3 and 2-3 planes respectively; the terms k_{13} and k_{23} are shear correction factors in the 1-3 and 2-3 planes respectively [11-14]. In the third-order and in the layerwise theories, the shear correction factors are not considered in the formulation.

Conclusions

This paper presents a summary of a few concepts that illustrate the various possible approaches to modelling of composite and sandwich laminates. As a recommendation, it can be stated that for thin composite laminates, single-layer theories (classical or shear deformation) are suitable for describing the kinematics, where for thick composite or sandwich laminates, three-dimensional or layerwise theories should be applied.

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