A MIXED FORMULATION FOR THE CLASSICAL PROBLEM OF THEODORE VON KARMAN: PURE BENDING OF CURVED TUBES.

(A Technical Note)

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ABSTRACT

This paper presents a hybrid formulation for the classical solution by Theodore von Karman on the problem of the uniform bending of curved tubes. His method consisted on assuming a trigonometric shape function approach for the deformed surface of the shell, having the calculation of the unknowns been carried out minimizing the total potential energy involved in the deformation process of the shell. The present solution deals with unknown membrane forces, following the longitudinal direction and unknown displacements for the ovalization of the curved pipe. The results are discussed and commented in the sequel.

1. INTRODUCTION

The problem of the stress and deformation analysis of curved tubes has no closed form solution starting from the equilibrium equations for shell analysis. As a matter of fact, the toroidal shell geometry does not allow a determinate set of equations for the stress analysis of such kind of piping accessories, unless a shape for the deformed surface is assumed prior the development of the equations for the problem set-up. Theodore von Karman (1911) presented an outstandingly simple and effective method to solve the here referred problem, having assumed a shape for the distorted structure followed of the principle of the minimum potential deformation energy. In the actual investigation, a great amount of solutions for the stress analysis for such kind of piping parts with generalised boundary conditions are based on many of the Von Karman assumptions, leading to reliable results. Some accurate solutions were, however, based on equilibrium equations and constitutive relations. This is, for the example, the method based on the thin shell theory from Novozhilov (1970), presented by J. F. Whatham and J. J. Thompson (1979) and J. F. Whatham (1986).
The study described here consists essentially in revisiting the solution of Von Karman in its original structure and presenting guidelines for more accurate results.

2. THE MIXED FORMULATION IN FINITE ELEMENT ANALYSIS

The mixed formulation consists in assuming as unknowns for a finite element problem, a set of forces, replacing some of the displacement vector, mixed with other set of displacements, which it is desirable to be kept in the formulation. A considerable amount of contribution has

appeared after the pioneer work of Theodore Pian *et al* (1983) in structural applications. Detailed research in this area, is also due to Herrmann (1968).
The mixed formulation may lead to a more straightforward definition of a stress field, without the need of post-processing routine scheme in a finite element analysis. Essentially, it leads to a more realistic formulation in some plastic analysis problems, where it is more important to assess directly the strain (or stress) variation of the structure, rather than the displacement field. This is, for example, the case of plastic forming, where the shape and time variation for the displacement field (if specified) is prescribed.
Considering a finite element where the nodal parameters are set-up as a mixed formulation, the typical structure for the problem has the following equation:

$$\begin{bmatrix} [A] & [E] \\ [E]^T & [(0)] \end{bmatrix} . \begin{Bmatrix} \mathbf{F} \\ \mathbf{U} \end{Bmatrix} = \begin{Bmatrix} \delta_{\mathbf{e}} \\ \mathbf{N_e} \end{Bmatrix} \tag{1}$$

The previous equation involves as unknowns, a set of displacements and forces (or stresses, if assumed), while the right hand side has the complementary parameter definition; this is, displacements for the force assigned equations ($\delta_{\mathbf{e}}$ to vector **F**) and forces to the displacement assigned equations ($\mathbf{N_e}$, to vector **U**). The set-up of the matrix form as (1) deals, not only with the elastic deformation energy, but also with the concept of *complementary energy*, or the *Hellinger-Reissner* variational formulation, clearly described by Den Hartog (1952). In the present analysis was followed the algorithm explained Zienkiewicz and Taylor (1992).

2.1 The solution of von Karman for a pure in-plane bending of a curved pipe

The shell analysis deals with equilibrium equations and constitutive relations. These results from the application of a differential operator, associated with an assumed deformation model, to a displacement vector. Defining the strain from the displacement vector, one can insert the so-obtained expression into the constitutive relations and completing the solution with the equilibrium equations. To reach a determinate expression, solving the problem for displacements or specific forces, one must solve a set of differential equations, eventually having a close form solution. A powerful and alternatives tool to solve practically all linear shell analysis problems consists on the application of the virtual work theorem, where a set of parameters associated to shape functions are used to define the displacement field. This was the main structure of the solution of von Karman (1911) to solve the problem of the pure bending of curved pipes.

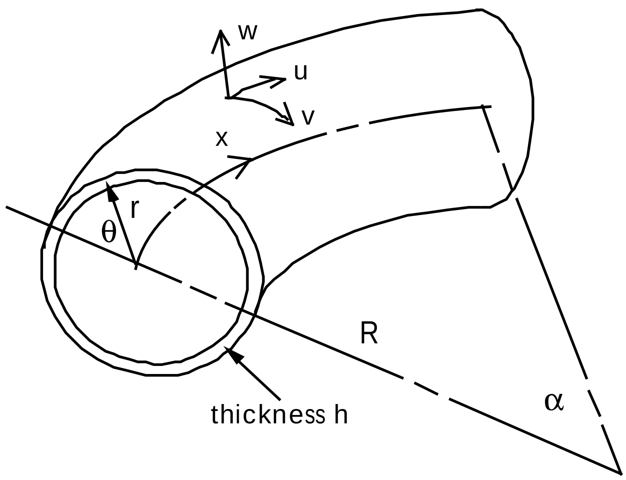

Figure 1: Geometry of a curved pipe as part of a toroidal shell and the shell displacement.

To base his solution on a set of physically realistic concepts, leading at the same time to the manipulations of simple expressions, the following assumptions are included in the von Karman solution:

a) The deformation for the curved pipe deals with a semi-membrane strain model. With this model, it is considered that the shell has membrane stiffness along both the meridional (θ) direction and the longitudinal (x) direction, wile there is bending stiffness only along the meridional direction, as a consequence of the ovalization.

b) The shell has meridional inextensibility; this means that hoop strain $\varepsilon_{\theta\theta} = 0$.

c) The mean curvature radius R is assumed much greater that r; this means that the factor $\rho = r/R = 0$, where it appears in the strain expressions further on.

d) There is no shear strain $\gamma_{x\theta}$ to be considered in the solution, as only pure bending is assumed.

The following expressions for the strains are as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{R}\left(w\cos\theta - v\,\mathrm{sen}\theta\right)$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\left(w + \frac{\partial v}{\partial \theta}\right) = 0 \text{ (assumption } \mathbf{b)} \text{ for inextensibility)}$$

$$k_{xx} = 0 \text{ (no curvature change along longitudinal direction)}$$

$$k_{\theta\theta} = \frac{1}{r^2}\left(-\frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2}\right) \text{ (meridional curvature variation, from ovalization)}$$

The solution of von Karman is based on the principle of virtual work, as described by J. P. Den Hartog (1952):

$$\frac{\partial U_t}{\partial a_i}\,\delta a_i = 0 \tag{2}$$

where U_t is the total energy involved in the elastic bending of the pipe and a_i is a set of unknown parameters defining the displacement field. The total energy is calculated from the expression:

$$U_t = \frac{1}{2}\int_V \varepsilon^T.D\varepsilon\ dv + M\delta\alpha,\ \text{ with } dv = hrd\theta dx \tag{3}$$

Where M is the bending moment and D is the elasticity matrix (for meridional inextensibility):

$$\frac{E\,h}{1-\nu^2}\begin{bmatrix} 1 & 0 \\ 0 & \dfrac{h^2}{12} \end{bmatrix} \tag{4}$$

ε is the strain vector, containing the following entities:

$$\varepsilon^T = \{\varepsilon_{xx},\, k_{\theta\theta}\} \tag{5}$$

which are the only strains from (1) contributing for strain energy. To set-up the deformation vector, it is necessary to assume a shape for the deformed surface of the shell. The three displacements u, v, w are configured as follows:

$$u = \frac{\delta\alpha}{L}.x$$

$$v = \sum_{i=1}^{n}\left\{-\frac{a_i}{n+1}\sin\,(n+1)\,\theta\right\} \tag{6}$$

$$w = \sum_{i=1}^{n}\{a_i\ \cos\,(n+1)\,\theta\}$$

where in the previous expressions, $\delta\alpha$ is the rotation of ends of the curved pipe and L is the mean arch length. The displacements v and w are defined form Fourier expansions, including the previously mentioned set of unknowns, a_i. The choice of trigonometric shape functions in v and w accomplishes the meridional inextensibility of the pipe (equation 1). Inserting expressions for the displacements (6) in the deformations (1) and calculating the total energy in (3), it is now possible to set-up the solution after the definition of the system of equations (2):

$$\begin{cases} \dfrac{\partial U}{\partial a_i} = 0 \\ \\ \dfrac{\partial U}{\partial \delta\alpha} = 0 \end{cases} \qquad (7)$$

In the previous systems, it is possible to suppose that the bending moment M is prescribed, appearing as unknowns the ovalization parameters a_i and the end rotation angle $\delta\alpha$. Alternatively, $\delta\alpha$ can be prescribed and (7) is reduced to the unknowns a_i, being M calculated from the expression for bending moment, which is uniform along the longitudinal arch.

$$M = \int_0^{2\pi} \sigma_x \, . \, h\, r^2 \cos\theta \, d\theta \qquad (8)$$

This procedure is detailed, for example, in works from Den Hartog (1952) and R. Kitching (1970). For a sake of simplicity and clarity of explanation, these two authors present the complete solution for the study of the curved pipe flexibility, having considered only one trigonometric term in the Fourier expansions (6). The following method, dealing with a mixed formulation for the problem discussed herein will consider the same simplification, in spite of the complete Fourier expansions for a more accurate solution being presented.

2.2 - The mixed formulation for the one-term semi-analytic solution in the uniform bending of a curved pipe

As shortly described in the previous sections, the mixed formulation in a finite element problems is condensed in a system of equations (Zienkiewicz and Taylor, (1992)), where it can be found as unknowns forces (or stresses) and displacement parameters. The present method deals with the following unknowns:

- $N_\theta(\theta)$, as membrane specific forces for the pure bending;
- **a**, as the displacement vector associated to the pipe ovalization;
- $\delta\alpha$, as the end pipe bending angle.

The ovalization displacement vector v and w, in (6) resumed to one trigonometric term only, is:

$$v = -\frac{a}{2}\,\mathrm{sen}\; 2\theta$$

$$w = a\,\cos\, 2\theta \qquad (6\text{-a})$$

$$u = -\frac{\delta\alpha}{L}\, . \, x$$

The reason for the minus sign in displacement u results from the fact that the negative rotation $\delta\alpha$ (counter-clockwise) gives a positive ovalization parameter, considering that θ is computed from pipe *extrados* along the clockwise direction as presents in fig. 1.
The substitution of parameters (6-a) in deformations in section 2.1, gives:

$$\varepsilon_{xx} = -\frac{\delta\alpha}{L}\, r\,\cos\theta + \frac{a}{R}\cos^3\theta$$

$$k_{\theta\theta} = \frac{1}{r^2}\,.\,3a\,\cos 2\theta \qquad (1\text{-a})$$

By observing the expression (1-a) for the membrane strain ε_{xx}, it is noted two trigonometric terms, respectively associated to shape functions $\cos\theta$ and $\cos^3\theta$, integrate the membrane strain. The best solution for N_θ will naturally contain both trigonometric terms, this ensuring the same accuracy, as does the von Karman solution. Therefore, it will be assumed that:

$$N_{xx} = \begin{bmatrix}\cos & , & \cos^3\end{bmatrix}\begin{Bmatrix}N_0\\ N_1\end{Bmatrix} = \mathbf{C}^T\ \mathbf{N}_{\theta e} \qquad (7)$$

As a consequence of the ovalization, a meridional bending moment is generated. The following constitutive relation is considered in the sequel:

$$m_\theta = \frac{E\,h^3}{12(1-\nu^2)}\,.\,k_{\theta\theta} \qquad (8)$$

This relation inserts one more unknown in the mixed formulation; however, it will be condensed, as shown next. The variation of the ovalization bending moment in (8) must follow naturally the same variation for $k_{\theta\theta}$. Therefore, it is assumed:

$$m_\theta = m_0\,\cos 2\theta \qquad \text{(for the \textit{one term} trigonometric solution)} \qquad (9)$$

The total energy for the mixed formulation in the present problem is:

$$\frac{1}{2}\int_0^L\int_0^{2\pi}\{C_\theta\ N_{\theta e}\}^T\,\frac{1-\nu^2}{E\,h}\left\{C_\theta^T\ N_{\theta e}\right\} r\,d\theta\,dx + \frac{1}{2}\int_0^L\int_0^{2\pi} m_0\,.\frac{12\left(1-\nu^2\right)}{E\,h^3}\,m_0\ r\,d\theta\,dx$$

$$-\int_0^L\int_0^{2\pi}\{C_\theta\ N_{\theta e}\}^T.\left[-\frac{\delta\alpha}{L}\,r\,\cos\theta + \frac{a}{R}\cos^3\theta\right] r\,d\theta\,dx$$

$$-\int_0^L\int_0^{2\pi}\{m_0\ \cos 2\theta\}\left[\frac{3a}{r^2}\cos 2\theta\right] r d\theta dx - M\delta\alpha = U_t \qquad (10)$$

In the previous expression, the first two integrals refer to the elastic energy from the membrane and bending forces assumed as unknowns. The last two integrals refer to the *complementary energy* (the *Hellinger-Reissner* complementary factors). They result from the deformation work produced by the unknowns N_0, N_1 and m_0 against the deformations arising from the assumed displacements $\delta\alpha$ and a. The reason of the negative sign for the complementary energy integrals results from the work of forces N_0 and N_1 being performed against deformations ε_{xx} and $k_{\theta\theta}$.

If the rotation $\delta\alpha$ is being prescribed, then the vector of unknowns for the present problem is:

$$X_e = \{N_0,\ N_1,\ m_0,\ a\} \tag{11}$$

Now the minimization of the total energy leads to the set-up of system (7) after simplification for the length L:

$$\begin{cases} C^{-1}\int_0^{2\pi}\cos^2\theta\, r\, d\theta\, N_0 + C^{-1}\int_0^{2\pi}\cos^4\theta\, r\, d\theta\, N_1 + 0 - \frac{1}{R}\int_0^{2\pi}\cos^4\theta\, r\, d\theta . a = -\frac{\delta\alpha}{L}\int_0^{2\pi}\cos^2\theta\, r^2\, d\theta \\ C^{-1}\int_0^{2\pi}\cos^4\theta\, r\, d\theta\, N_0 + C^{-1}\int_0^{2\pi}\cos^6\theta\, r\, d\theta . N_1 + 0 - \frac{1}{R}\int_0^{2\pi}\cos^6\theta\, r\, d\theta . a = -\frac{\delta\alpha}{L}\int_0^{2\pi}\cos^4\theta\, r^2\, d\theta \\ 0 \quad + 0 \quad + D^{-1}\pi r\, m_0 - \frac{3\pi}{r} a = 0 \\ -\frac{1}{R}\int_0^{2\pi}\cos^4\theta\, r\, d\theta\ N_0 - \frac{1}{R}\int_0^{2\pi}\cos^6\theta\, r\, d\theta .\ N_1 - \frac{3\pi}{r} m_0 + 0 = 0 \end{cases} \tag{12}$$

where $C^{-1} = \dfrac{1-\nu^2}{E h}$ and $D^{-1} = \dfrac{12\left(1-\nu^2\right)}{E h^3}$.

The bending moment M does not appear in the previous system, as line and column referred to parameter $\delta\alpha$ are eliminated.

The parameter m_0 is easily condensed from the third equation.

Solving for N_0 and N_1, one gets easily the longitudinal membrane stress σ_{xx}, just simply dividing N_x (in (7)) by the thickness h. The ovalization parameter, a, completes the stress field definition for the calculation of $k_{\theta\theta}$ or in (9) for m_θ.

As an example, consider the pure in plane bending of a curved pipe, having the following dimensions and mechanical properties (see fig. 1)

r = 50 mm
R = 200 mm
L = 157.0796 mm
E = 210 GPa
ν = 0,3

The pipe has both ends free to ovalize. The section rotation is supposed to start from zero at one end, increasing uniformly until the valve of $\left\{\frac{\delta\alpha}{L} \, . \, x\right\}_{x=L} = \delta\alpha$.

The previous solution gives:

$\mathbf{a} = 74.111.\,\delta\alpha$
$N_0 = -\,0.3183 C\,\delta\alpha$
$N_1 = +\,0.370531 C\,\delta\alpha$

$$N_x = \left(N_0 \, . \cos\theta + N_1 \cos^3\theta\right) \times C\delta\alpha$$

The result for the ovalization parameter a, coincides with the one calculated with the totally assumed displacement solution from von Karman. The calculation of the stresses is carried out readily when compared with the method of von Karman, in spite of demanding more parameters for the solution. Figure 2 shows the graphical variation of the membrane effort N_x from $\theta = 0$ (*extrados*) until $\theta = 180^0$ (*intrados*), taken at any transverse section of the pipe. The values coincide with the corresponding after the solution of von Karman, as reported by Kitching (1970).

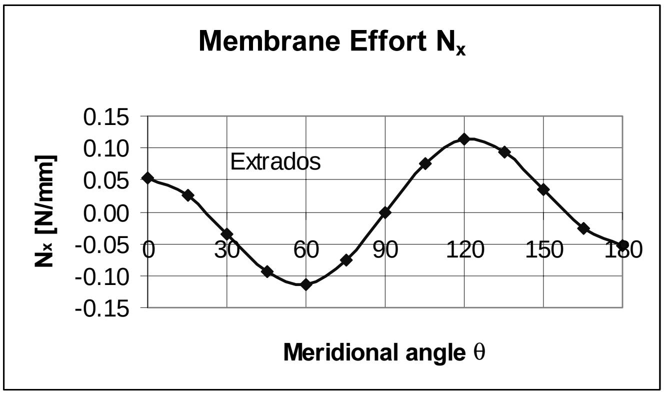

Figure 2: Values for longitudinal membrane effort $N_x \times C\delta\alpha$ for a uniform bending of a curved pipe.

2.3 - A more accurate solution for the mixed formulation of the von-Karman problem.

In the previous section, the basic solution involving a mixed formulation technique (one trigonometric term for the section ovalization), was presented. Naturally, a more exact solution is achieved considering more trigonometric terms for the section ovalization. As it was referred, the expression for the longitudinal strain carries a first order trigonometric function, cosθ, followed of the terms wcosθ-vsinθ. Having assumed the transverse inextensibility for the pipe, an effective shape function of order *n* for the ovalization, would be:

$$SC_n = \cos\theta \cos n\theta + \frac{1}{n}\sin\theta \sin n\theta, \quad (\text{with } n = 2,3,...) \tag{13}$$

The complete trigonometric shape function matrix is now written as:

$$[\mathbf{N}]^T = [\cos\theta, \mathbf{SC}^T] \tag{14}$$

where $\mathbf{SC}^T$ contains the n generic terms in (13). Naturally, n parameters a_n and m_n, respectively for the displacement and bending moment dues to the ovalization, must be considered. The updated displacement vector is:

$$\mathbf{X}_e=\{N_0, \mathbf{N}_{sc}, \mathbf{m}, \mathbf{a}\} \tag{15}$$

where:
$\mathbf{N}_{sc} = \{\mathbf{N}_{sc2}, \mathbf{N}_{sc3}, \ldots\}$, $\mathbf{m} = \{m_2, m_3, \ldots\}$ and $\mathbf{a} = \{a_2, a_3, \ldots\}$

3 - CONCLUSIONS

The solution present here, though not leading to a more economic and straightforward analysis than the one involving a totally assumed displacement, revealed itself quite accurate. In more complex problems, dealing with the stress analysis of these accessories loaded under plastic conditions, the mixed formulation may set-up more useful tools, as the force variation (as unknowns) depending on the geometric and/or material non-linearity, is calculated without the need of additional algorithms.

4 - REFERENCES

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